# A generalized closed-form solution for 3D registration of two-point sets under isotropic and anisotropic scaling 

Maria Chatrasingh ${ }^{\text {a }}$, Cholatip Wiratkapun ${ }^{\text {b }}$, Jackrit Suthakorn ${ }^{\text {a,* }}$<br>${ }^{a}$ Department of Biomedical Engineering, Center for Biomedical and Robotics Technology, Faculty of Engineering, Mahidol University, Salaya, Thailand<br>${ }^{\mathrm{b}}$ Department of Radiology, Faculty of Medicine, Ramathibodi Hospital, Mahidol University, Thailand

## ARTICLE INFO

## Keywords:

Anisotropic Scaling
Non-uniform scaling
Point set registration
3D registration
Ultrasound images


#### Abstract

Fitting of two-point sets under non-uniform scaling involves several applications, including ultrasound-guided surgery, where ultrasound images generally show anisotropy between its lateral and depth resolution. However, the current state-of-the-art only accomplishes the closed-form solution under the assumption of isotropic scaling, and only the iterative form of the solution is available for registration regardless of scaling type. A generalized closed-form solution, including anisotropic scaling, could robust the computation in the applications of multi-modality image fusion and surgical navigation based on ultrasound images. In this paper, a generalized closed-form solution (GCFS) that accounts for isotropic scaling and anisotropic scaling is proposed. GCFS finds the least-squares solution of rotation, translation, and scaling based on the singular value decomposition (SVD). The method was demonstrated to 1) register two 3D point sets under the effect of anisotropic scaling and 2) register two coplanar 3D point sets under the effect of anisotropic scaling. This study evaluates 3D registration between the preoperative model and intraoperative ultrasonic scanning of a femur and 2D registration of ultrasound images to the phantom for ultrasound calibration. In addition, the conditions that affect the transformation estimations like estimating reflection and noise degeneracy were also considered in the evaluation.


## Introduction

Point set registration is essential for image-guided surgery since guidance requires rigid mapping between image coordinates and world coordinates where the surgery takes place or mapping among multimodality imaging involved in the scene [1]. The determination of a transformation between any two-point sets in different coordinate frames is based on minimizing the registration error using the equation (1),
$F=\|A-H S B\|^{2}$
where $A$ and $B$ are the set of corresponding position vectors in two different coordinate frames, $F$ is the registration error corresponding to the registration via the transformation $H$, and $S . H$ is the special Euclidean group, denoted by SE(n), composed of rotation and translation components. $S$ is the diagonal matrix where each diagonal component is related to the scaling factor of an individual axis. The scaling factors, as denoted, could be non-uniform among axes. Nonuniform or anisotropic scaling commonly occurs when the inherited
device acquires spatial information using different principals for different measuring axes. In image-guided surgery, this often happens with ultrasound modality. Ultrasound imaging has lateral resolution depending on the design of the transducer array, while depth resolution depends on the speed of sound waves and wave frequency [2]. Even though ultrasound machines by default were configured for uniform scaling based on average propagation speed in soft tissue, the speed could be varied, causing anisotropic scaling by nature [2,3].

Ultrasound-guided surgery has been widely adopted in either commercialized systems or research fields because of its prominent features in freehand use and real-time imaging [4,5]. Ultrasound is susceptible to poor resolution, signal-to-noise ratios (SNR), and limited field-of-view; therefore, it is often used in conjunction with other imaging volumes, such as CT and MRI, or a navigation system [6]. GE Healthcare recently launched the LOGIQ E9, a new ultrasound fusion system, which fuses ultrasound images with preoperative CT/MRI and displays CT or MR slices corresponding to probe motion [7,8]. The ultrasound-guided liver biopsy system adopts the navigation capability so that features in an ultrasound image could be related to the biopsy

[^0]needle in world coordinates [9]. Both examples of ultrasound-based applications illustrate the necessity for the registration involving anisotropic scaling with high precision and fast computation.

Only a few techniques have been used to estimate the spatial transformation in anisotropic registration [10]. To the best of the author's knowledge, none of those existing have presented closed-form solutions for determining rotation and scale that truly minimize registration errors. The current development of point set registration techniques are mostly based on the standard least squares solutions for fitting two-point sets, which are either the closed-form solution based on singular value decomposition proposed by Arun et al. [11] or the quaternions solution proposed by Horn [12]. Note that both solutions only account for uniform scaling. Horn has also discussed the extension of his solution toward non-uniform scaling. However, the rotation component is still derived based on a uniform scaling assumption. As a result, the overall solution does not minimize the registration errors. An interesting iterative approach to registration point sets with non-uniform scaling is the block relaxation (BR) technique proposed by Gower and Dijksterhuis [13]. The technique iteratively determines scale while fixing rotation and determines rotation while fixing the scale until reaching the minimum error. However, it has serious limitations to ensure convergence to a minimum [14]. Dosse and Berge et al. proposed a technique of orthogonal procrustes analysis for anisotropic scaling and its generalized form [15], which is based on the BR technique but with the guarantee of iterative convergence.

To further extend the solutions toward non-corresponding point cloud registration, the modified Iterative Closest Point (ICP) [16-18] registers point clouds under anisotropic scaling conditions by iteratively reducing the errors. Chen et al. [17] did guarantee the convergence of their anisotropic ICP algorithm toward a local solution; however, there is concern about global minima. As far as can be confirmed, the only closed-form solution for registration involving anisotropic scaling has been proposed by Qu et al. [19]. Qu et al. derived the solution by transforming the point set into the space of unit covariance and determined the rotation under the condition of unit scaling. These components are then transformed back into the generalized covariance space, where spatial transformation includes rotation, scale, and shear. Since Qu's solution was not based on the minimization of the objective function, as previously mentioned, it is impossible to exclude the shearing factors from the solution. The shear factor in the solution always gives the best fit between the registration point sets, even with the data containing noise. The derived transformation might not represent true spatial mapping among image modalities or objects in world space in the case where shearing has no physical meaning, i.e., preoperative calibration.

We propose a closed-form solution to registration point sets regardless of scaling type, which will be called a generalized closed-form solution (GCFS) for the rest of this paper. The method derives the least squares solution based on the minimization of registration errors concerning the rotation and scale (without shear). Singular value decomposition (SVD) was used to extract the rotation components, followed by diagonal factorizing of the scale matrix. As mentioned previously, the applications of ultrasound registration were used to evaluate the proposed solution. The focus of this paper is to compare the proposed solution to the other closed-form solutions currently in practical use.

The construction of this paper is as follows. First, a brief explanation of the traditional least squares solution for isotropic scaling registration was stated. Then, GCFS was derived according to minimizing the registration objective function. Lastly, the evaluation of the solution toward ultrasound registration in either 3D or 2D was verified in comparison with other available closed-form methodologies. The evaluation was based on simulation registration under a level of noise-added data where the solution could easily degenerate.

## Materials and methods

Traditional least squares solution for fitting two point sets with isotropic scale

The problem of estimating a transformation between two matrices is known as the Orthogonal Procrustes Problem [19]. With the assumption of isotropic scaling and rigid transformation, the least squares solution, as proposed by Arun et al. [11] and Umeyama [20], is explained as follows.

Given two point sets with zero centroids; $\widehat{A}$ and $\widehat{B}$ with the same number of points and with known correspondence, under the assumption of isotropic scaling, the equation of registration error in Eq.(1) could be simplified to
$F=\|\widehat{A}-\lambda R \widehat{B}\|^{2}$
where $R$ is a rotation matrix and $\lambda$ is a scalar factor for the isotropic scaling matrix ( $\lambda=1$ for non-scaling conditions). The registration problem involves finding $R$, and $\lambda$ that minimizes $F$. The solution is given as
$R=\ddot{U} D \ddot{V}^{T}$
$D=\left\{\begin{array}{cc}I & \text { if } \operatorname{det}(U) \operatorname{det}(V)=1 \\ \operatorname{diag}(1,1, \cdots, 1,-1) & \text { if } \operatorname{det}(U) \operatorname{det}(V)=-1\end{array}\right.$
and $\widehat{A} \widehat{B}^{T}=\dddot{U} \ddot{\Sigma} \ddot{V}^{T}$ is the singular value decomposition of $\widehat{A} \widehat{B}^{T}$.
The scalar factor of the scaling matrix is estimated according to [14] as
$\lambda=\frac{\operatorname{tr}(\ddot{\Sigma})}{\|\widehat{B}\|^{2}}$

## Propose generalized Closed-Form solution (GCFS) for fitting two point sets regardless type of scale

To generalize the scaling types, the replacement of the isotropic scaling factor $\lambda$ in Eq. (6) by a diagonal matrix $S$ obtains
$F=\|\widehat{A}-R S \widehat{B}\|^{2}$
Lemma: Let $\widehat{A}$ and $\widehat{B}$ be $m \times n$ matrices containing position vectors with zero centroids, $R$ is a $m \times m$ rotation (orthogonal) matrix, $S$ is a $m \times$ $m$ scaling (diagonal with positive entries) matrix, and $U \Sigma V^{T}$ is the singular value decomposition of $\widehat{A} \widehat{B}^{T}(\widehat{B} \widehat{B})^{-1} . R$ and $S$ that minimize $F$ are obtained with
$R=U D V^{T}$
$D=\left\{\begin{array}{cl}I & \text { if } \operatorname{det}(U) \operatorname{det}(V)=1 \\ \operatorname{diag}(1,1, \cdots, 1,-1) & \text { if } \operatorname{det}(U) \operatorname{det}(V)=-1\end{array}\right.$
and $S=\operatorname{diag}\left(s_{1}, s_{2}, \ldots, s_{m}\right)(9)$
$s_{j}=\frac{\operatorname{tr}\left(\widehat{A}^{T} R \Lambda_{j} \widehat{B}\right)}{\operatorname{tr}\left(\widehat{B}^{T} \Lambda_{j} \widehat{B}\right)}$
Giving $\Lambda_{j}$ a diagonal matrix of size $m \times m$ where all entities are zero except the diagonal entities $j$ is one.

Proof of Lemma: Given
$F=\|\widehat{A}-R S \widehat{B}\|^{2}=\|\widehat{A}\|^{2}+\|R S \widehat{B}\|^{2}-2 \operatorname{tr}\left(R S \widehat{A} \widehat{B}^{T}\right)$
minimization $F$ is determined by partial differentiation $F$ concerning RS
$\frac{\partial F}{\partial(R S)}=2 R S \widehat{B} \widehat{B}^{T}-2 \widehat{A} \widehat{B}^{T}=0$
Eq. (12) becomes,
$R S-\widehat{A} \widehat{B}^{T}\left(\widehat{B} \widehat{B}^{T}\right)^{-1}=0$
Note that Eq. (13) is the partial differentiation of $F$ for $R S$ intentionally to find $R S$ that minimized $F$ and later extracted the solution of $R$ and $S$ from $R S$, giving the constraints for each of them. Before directly giving the constraints of $R$ and $S$ as they are, we give $L \equiv S$ and define $L$ as a positive definite matrix instead of $S$, the diagonal matrix. Since $S$ is a diagonal matrix with all positive entries, it is one of the positive definite matrices. $L$ covers the wider range of multiple positive definite matrices, such as shearing, symmetrical distortion, and scaling matrices in a generalized registration. $S$ is defined as a maximized diagonal matrix extracted from $L$. $L$ and $R$ are constrained as the positive definite matrix and an orthogonal matrix, respectively, as follows.

Note that with SVD of $\widehat{A} \widehat{B}^{T}\left(\widehat{B} \widehat{B}^{T}\right)^{-1}=U \Sigma V^{T}$ Eq. (13) becomes,
$R L=\widehat{A} \widehat{B}^{T}\left(\widehat{B} \widehat{B}^{T}\right)^{-1}$
Multiply Eq. (14) with the transpose of itself gives,
$L^{2}=V \Sigma^{2} V^{T}$
Since $L$ it is a positive definite symmetric matrix, the inverse of $L$ determining from Eq. (15) is according to [21] where,
$L^{-1}=V \Sigma^{-1} D V^{T}$
From Eq. (14) and Eq. (16), we could determine,
$R=\widehat{A} \widehat{B}^{T}\left(\widehat{B} \widehat{B}^{T}\right)^{-1} L^{-1}=U \Sigma V^{T} V \Sigma^{-1} D V^{T}=U D V^{T}$
To determine the scaling matrix, the objective function $F$ is minimized for $S$. $S$ is a diagonal matrix with all positive entries. Partial differentiation of $F$ for $S$ gives,
$\frac{\partial F}{\partial S}=\sum_{j=1}^{m}\left(2 \operatorname{tr}\left(\widehat{B}^{T} S \Lambda_{j} \widehat{B}\right)-2 \operatorname{tr}\left(\widehat{A}^{T} R \Lambda_{j} \widehat{B}\right)\right)=0$
With the restriction that each $2 \operatorname{tr}\left(\widehat{B}^{T} S \Lambda_{j} \widehat{B}\right)-2 \operatorname{tr}\left(\widehat{A}^{T} R \Lambda_{j} \widehat{B}\right)=0$,
$S=\operatorname{diag}\left(s_{1}, s_{2}, \ldots, s_{m}\right)$

## Where

$s_{j}=\frac{\operatorname{tr}\left(\widehat{A}^{T} R \Lambda_{j} \widehat{B}\right)}{\operatorname{tr}\left(\widehat{B}^{T} \Lambda_{j} \widehat{B}\right)}$
Giving $\Lambda_{j}$ a diagonal matrix of size $m \times m$ where all entities are zero except the diagonal entities $j$ is one. Note that the constraint of L as a positive definite matrix gives the solution in Eq. (7) and Eq. (9) true in the more generalized registration equation involving other components such as shearing or symmetrical distortion, although that is not the focus of this paper. The proof of Lemma shows that it is true in the boundary of mathematical proof, and it gives the solution for $R S$ minimized the registration error, not the $R$ that minimized the error.

## Reduced form of GCFS in isotropic scaling or Non-Scaling conditions

The use of the solution in Section 2.2 is not limited only to the conditions of anisotropic scaling. For isotropic scaling or non-scaling conditions, the solution is feasible, as shown below.
$F=\|\widehat{A}-R S \widehat{B}\|^{2}$ where $S=\lambda I$
According to the same proof of Eq. (12), $S \widehat{B} \widehat{B}^{T}$ is a positive definite
symmetric matrix when $S=\lambda I$. Therefore, we give $L=S \widehat{B} \widehat{B}^{T}$ and define $L$ it as a positive definite symmetric matrix. Instead, Eq.(14) is derived for isotropic scaling as,
$R L=\widehat{A} \widehat{B}^{T}$
Then, given
$\widehat{A} \widehat{B}^{T}=\ddot{U} \ddot{\Sigma} \ddot{V}^{T}$
according to GCFS in Eq.(17), the rotation matrix becomes
$R=\ddot{U} D \ddot{V}^{T}$
which, according to Eq. (20) and Eq.(24),
$s_{j}=\frac{\operatorname{tr}\left(A^{T} R \Lambda_{j} B\right)}{\operatorname{tr}\left(B^{T} \Lambda_{j} B\right)}=\frac{\operatorname{tr}\left(\widehat{A} B T \Lambda_{j} R^{T}\right)}{\|\widehat{B}\|^{2}}=\frac{\operatorname{tr}\left(\ddot{U} \ddot{\Sigma}^{T} \ddot{V}^{T} \Lambda_{j} \ddot{V} D \ddot{U}^{T}\right)}{\|\widehat{B}\|^{2}}=\frac{\operatorname{tr}(\ddot{\Sigma})}{\|\widehat{B}\|^{2}}$
For isotropic scale $s_{1}=s_{2}=\ldots .=s_{m}=\lambda$, therefore
$\lambda=\frac{\operatorname{tr}(\ddot{\Sigma})}{\|B\|^{2}}$
Eq.(24) and Eq.(26) are the same as proposed by [11] and [14] for a singular value estimation of rotation for isotropic scaling or non-scaling ( $\lambda=1$ ). Therefore, GCFS is the generalized form of traditional least squares solution $[11,20]$ that involves non-uniform scaling.

## Special Case: Register coplanar point sets (Rank $=2$ ) in 3D space

The common scenario to estimate the transformation between twopoint sets is fitting a 2D image into a plane in 3D models, as depicted by the application of ultrasound and CT fusion [13]. The individual point sets $\widehat{A}$ and $\widehat{B}$ are coplanar but non-collinear. Since both present zero variance in one of the three dimensions $\widehat{A} \widehat{B}^{T}$ and $\widehat{B} \widehat{B}^{T}$, as in Eq.(17), have a zero eigenvalue, that is, the singular value decomposition of $\widehat{A} \widehat{B}^{T}$ $\widehat{A} \widehat{B}^{T}=U \Lambda V^{T}$
and the singular value decomposition of $\widehat{B} \widehat{B}^{T}$
$\widehat{B} \widehat{B}^{T}=\widetilde{U} \widetilde{\Sigma} \widetilde{U}^{T}$
where $\Lambda=\operatorname{diag}\left(\lambda_{i}\right), \lambda_{1} \geq \lambda_{2} \geq \lambda_{3}$ and $\widetilde{\Sigma}=\operatorname{diag}\left(\sigma_{i}\right), \sigma_{1} \geq \sigma_{2} \geq \sigma_{3}$ have zero eigenvalues $\lambda_{3}=\sigma_{3}=0$. When the matrices are singular, the solution in Eq. (17) is indefinable. This amounts to rank deficiency and infinite solutions for determining a transformation that minimizes registration errors.

For this, the solutions are determined separately for the two scenarios.

1) $\widehat{B}$ is a coplanar point set and is 2D data, e.g. $b_{i}=\left[\begin{array}{lll}x & y & 0\end{array}\right]^{T}$, where $\widehat{A}$ is a coplanar point set in 3D space $a_{i}=\left[\begin{array}{lll}x & y & z\end{array}\right]^{T}$, such as registering a 2D image into a known plane in a 3D image volume. In this case, $s_{i}$, where $i$ corresponds to the dimension which is the rank deficient, could not be retrieved as $\lambda_{i}=\sigma_{i}=0$ and both $\widehat{A} \widehat{B}^{T}$ and $\widehat{B} \widehat{B}^{T}$ are singular matrices. This singularity could be avoided by giving $s_{i}=1$ and giving $\lambda_{i}=\sigma_{i}=1$ before calculating $\widehat{A} \widehat{B}^{T}\left(\widehat{B} \widehat{B}^{T}\right)^{-1}$ in Eq. (17).
2) Both $\widehat{A}$ and $\widehat{B}$ are coplanar point sets and are 3D data; e.g., $a_{i}=$ $\left[\begin{array}{lll}x & y & z\end{array}\right]^{T}$ and $b_{i}=\left[\begin{array}{lll}x & y & z\end{array}\right]^{T}$. The singularity in Eq. (17) could be avoided by dimension reduction. However, the missing dimensionality results in unknown parameters; one rotation component corresponds to a scaling factor. Assigning the value to one scaling factor,


Fig. 1. Comparisons of noise levels and (a) fiducial registration error (FRE) (b) relative error ( $\varepsilon$ ) using GCFS, Qu' solution, and the traditional least squares solution.
all unknown rotation components could be retrieved via Eigenvalue decomposition.

## Results

The performance of GCFS to estimate a transformation under the generalized scaling condition was evaluated with a set of synthetic data. In the evaluation, the scaling factors are randomly chosen therefore, the type of scaling could be either isotropic or anisotropic. The verification was performed on 1) registration of pair 3D point sets 2 ) registration of two coplanar point sets in 3D space. The implementation and calculation were based on a ( 2.40 GHz , Intel(R) Core (TM) i5 CPU) computer with MATLAB 2019 (The MathWorks, Inc., Natick, Massachusetts, United States) and its built-in singular value decomposition library.

Registration of 3D point sets (Rank $=3$ )
The authors applied a scenario of Computer-assisted Hip Resurfacing surgery [22] to evaluate GCFS. In Hip Resurfacing surgery, intraoperative position and orientation of femoral heads and tools are very important to drill and ensure the implant position. During surgery, the computer-assisted system would acquire 3D ultrasound scans of the bone surface. The bone surface will then be registered to a preoperative bone model, followed by the update of preoperative planning according to the actual bone position and orientation. Due to variations in ultrasound machines, the scanning bone surface would be scaled differently, and the scaling is likely to be non-uniform. Registration of the point sets acquired from scanning a proximal femur was used to evaluate the closed-form solution. Note that the point sets are the synthesis data. Ten linear transformations were randomly selected to fit the point set $\widehat{A}$ into another point set $\widehat{B}$. The random parameters are in the range as follows; $R_{x}, R_{y}, R_{z} \in[0, \pi], t_{x}, t_{y}, t_{z} \in[-2,2]$, and $s_{1}, s_{2}, s_{3} \in[0.3,1]$.


Fig. 2. The relative error of estimating $\widehat{H}=\widehat{R} \widehat{S}$ with the increasing level of noise.


Result After Registration (Example) At Various Noise Degeneracies
Fig. 3. Surface scan of a femur under the effect of anisotropic scaling registered to its own 3D model using GCFS at various standard deviations of Gaussian noise.


Fig. 4. The result of anisotropic scaling 3D point sets registration using closedform solution under Gaussian noise at various standard deviations.

Isotropic Gaussian noise with relative variance $\sigma \in[0,0.07]$ for the normalized data was added $\widehat{B}$ before the estimation of the transformation. GCFS was compared to the traditional least squares solution for isotropic scaling $[11,20]$ and Qu's closed-form method for general affine registration [19]. Since the principle to estimate the translation for all three methods is the same, the relative errors are evaluated only on the rotation and scale components. Fiducial registration errors (FRE); $F R E=\frac{1}{N} F=\frac{1}{N}\|\widehat{A}-R S \widehat{B}\|^{2}$ are calculated for each solution at each level


Fig. 5. Evaluation of GCFS in coplanar registration with the application in ultrasound calibration (a) an ultrasound image that contains the feature of the phantom and (b) a 2D alignment phantom.
of noise degeneracy to evaluate the tolerance to degeneracy [23]. In addition, the deviation from the identity matrix was evaluated to measure the relative error between the estimated matrix and the synthetic one. Let $\widehat{H}=\widehat{R} \widehat{S}$ be the product of multiplication between the synthetic rotation matrix and scaling matrix. $H$ is the $3 \times 3$ transformation matrix determined from the methods. $\varepsilon=\left\|I-\widehat{H} H^{-1}\right\|_{F}$ is the relative error between the true and estimated transformations. $\|\bullet\|_{F}$ denotes the Frobenius norm of the matrix.

Average FRE and $\varepsilon$ at various levels of noise is shown in Fig. 1, and Fig. 2 shows the results of the relative error of estimating $\widehat{H}=\widehat{R} \widehat{S}$ with the increasing level of noise. Fig. 3 depicts an example of the registration under anisotropic scaling using GCFS. The result of anisotropic scaling 3D point sets registration using a closed-form solution under Gaussian noise at various standard deviations is shown in Fig. 4.

As shown in Fig. 1(a), the traditional least squares solution shows a certain amount of FRE even at zero noise degeneracy, while GCFS and Qu's method [19] show none. Since both are based on the same least squares method, GCFS, and the traditional least squares method seem to display a linear relationship as noise increases. Qu's method, on the other hand, has FRE increasing rate greater than the other two methods as noise increases. Moreover, the efficiency of Qu's method toward estimating the true transformation (synthesis), as depicted in $\varepsilon$ (Fig. 1

Table 1
FRE, $\varepsilon$, and scaling factors of the ultrasound image determined using GCFS with ultrasound image mapping in 3D space (number of ultrasound images $n=20$ ).

|  | Mean | Standard Deviation | Maximum Deviation |
| :--- | :--- | :--- | :--- |
| FRE | 0.36 | 0.24 | 0.21 |
| $\varepsilon$ (Only rotation and scale) | 0.33 | 0.43 | 0.81 |
| Lateral scale factor | 1.23 | 0.21 | 0.64 |
| Depth scale factor | 1.11 | 0.17 | 0.78 |

(b)) is significantly worse compared with the others. The relative errors grow dramatically while GCFS and traditional least squares method show low increasing rates along the increasing of degeneracy. Among the three, GCFS presents the lowest FRE and $\varepsilon$ either at the ideal zero noise or under high degeneracy.

Registration of two coplanar point sets in 3D space $($ Rank $=2)$

Registration of the features visible in 2D ultrasound images with the corresponding phantom construction is one of the foundations of ultrasound calibration. The design of a 2D alignment phantom [24], laser cut on a 2 mm acrylic sheet (Fig. 5(b)), was used as the ultrasound calibration phantom in our experiment. The authors manually aligned the phantom plane to coincide with the scanning plane so the ultrasound images fully display the phantom, as illustrated in Fig. 5(a). Seven points from the vertices of the phantom were fitted to the corresponding points in the ultrasound images. Twenty ultrasound images, which perfectly capture the plane of the phantom, were selected. Ten random linear
transformations were selected as the frame transformation of the phantom in 3D space. The random parameters are in the range as follows; $R_{x}, R_{y}, R_{z} \in[0, \pi], t_{x}, t_{y}, t_{z} \in[-2,2], s_{1}, s_{2} \in[0.3,1]$, and $s_{3}=1$ The scale factors were not simulated but rather used as evaluated criteria. The efficiency of the 2D to 3D registration was evaluated based on the mean, standard deviation, and maximal deviation of FRE and the scale factors. The results of 2D to 3D registration is shown in Table 1.

A 2D point set acquired from a plane of 3D Parasaurolophus was used to evaluate the solution. This point set aligns on an $x-y$ plane of a 3D space with a constant $z$ value at 0 . Ten random linear transformations were selected to transform the point set $A^{\prime}$ into another point set, B. The result of the registration is shown in Fig. 6.

Table 1 indicates the mean, standard deviation, and maximum deviation of fitting the ultrasound features to the corresponding construction in 3D space. The average FRE and $\varepsilon$ are small, thus illustrating a close fit with the estimated transformation. For the scaling factors the true scaling factors were not known for the tested ultrasound machine. Since we expected certain scale factors for the specific probe and specific scanning medium, the deviation of the estimated scale would demonstrate the reliability of the solution. Although the results in Table 1 show a small standard deviation, as expected, the maximum deviations are relatively high. This might be the consequence of the manual alignment of the phantom plane with the scanning plane. However, further experiments must be performed to ensure the conclusion. Using the 2D phantom with symmetric shape, the reflections in the solutions were not found among the three methods.

Though the problem of anisotropic registration involves several


Before registration


$$
\sigma=0.0278
$$


$\sigma=0.0389$
$\sigma=0.05$

After registration
Fig. 6. The result of registering 3D coplanar point sets registration using closed-form solution under Gaussian noise at various standard deviations.
applications, this paper is motivated by the application of Ultrasoundguided surgery. Two examples of image-guided surgery involving ultrasound were chosen to evaluate this method. However, the registration between different modalities other than ultrasound could also be achieved with this generalized closed-form solution. The other medical applications which could benefit include surface reconstruction of organs in laparoscopic surgery with various optical localizer measurements [25,26]. Also, theoretically, the proposed solution could be used in other coordinate-registration applications that involve the component of shearing and symmetrical distortion, such as image registration [3,27-29], camera/image modalities calibration [30-32], optical tracking [33,34], computer vision [35,36], robot navigation [37,38], and augmented reality $[39,40]$ where the point set data are acquired from stereovision, 3D scanning, or an array of rangefinders. This testing with the dataset from the applications would need further exploration, and the comparison of the proposed method to the iterative method in terms of computational cost, accuracy, and stability will be focused on in future work.

## Conclusions

In this paper, we present a generalized closed-form solution (GCFS) for point sets fitting and pose estimation under isotropic or anisotropic scaling. The method uses singular value decomposition and covariance matrix to solve for the least squares solution. The method provides a generalized solution regardless of isotropic or anisotropic scaling. Note that under isotropic scaling, the solution reduces the form into the traditional least squares solution [5]. The mathematical proof of the solution, as well as the demonstration of the efficiency, are provided. The method was demonstrated to 1 ) register two 3D point sets under the effect of anisotropic scaling and 2) register two coplanar 3D point sets under the effect of anisotropic scaling. The accuracy of the method was evaluated in both cases under noise degeneracy. The estimated spatial transformations matched with the synthetic matrices and depicted the most efficient among all the methods. No instability was found during the evaluation. With this closed-form solution, the registration between different modalities which exhibit anisotropic scale could be achieved. The further extension of the closed-form solution such as registration of point clouds with unequal number of point member or unknown correspondence. The proposed solution could be used in the applications
such as image registration, camera/image modalities calibration, optical tracking, computer vision, robot navigation, and augmented reality where the point set data are acquired from stereovision, 3D scanning, or an array of rangefinders.

## CRediT authorship contribution statement

Maria Chatrasingh: Conceptualization, Methodology, Investigation, Data curation, Formal Analysis, Software, Writing - original draft. Cholatip Wiratkapun: Conceptualization, Methodology, Data curation, Resources, Validation, Visualization. Jackrit Suthakorn: Conceptualization, Methodology, Supervision, Funding acquisition, Validation, Visualization, Project Administration, Writing - review \& editing.

## Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

Data will be made available on request.

## Acknowledgments

The first author is supported by the Mahidol University Ph.D.-M.D. program. The ultrasound machine and experimental space are supported by Ramathibodi Hospital, Mahidol University. The authors would like to thank Dr. Branesh M. Pillai for assisting in manuscript drafting and editing support.

Funding statement.
This project is supported by the Reinventing University System through Mahidol University (IO 864102063000), and Research Funding through Mahidol University under the project: Integration of Surgical Navigation and Surgical Robotics for Breast Biopsy in Breast Cancer Using Mammogram and Ultrasound Images on Breast Mathematical Model (Pre-Clinic and Clinic Evaluation Phase for Product Commercialization).

Appendix 1:. The simple MATLAB code describe the calculation of the solution for 3D point sets

```
function [translation,rotation,scale]=genealized_method(q_A,q_B).
centroid_q_B =sum(q_B,2)/size(q_B,2);
centroid_q_A =sum(q_A,2) /size(q_A, 2);
q_B_center =q_B-repmat (centroid_q_B, [1, size(q_B, 2)]);
q_A_center =q_A-repmat (centroid_q_A, [1, size(q_A,2)]);
H =q_A_center*q_B_center';
P =q_B_center*q_B_center';
[UH,SH,VH]=svd(H);
[UP,SP,VP]=svd(P);
H_d= UH*SH*VH';
P_d= UP*SP*VP';
K_d=H_d*(P_d)^(-1);
[U,S,V]=svd(K_d);
s1=trace(q_A_center'*rotation*diag([1,0,0])*q_B_center)/trace(q_B_center'*diag([1,0,0])*q_B_center);
s2=trace(q_A_center'*rotation*diag([0,1,0])*q_B_center)/trace(q_B_center'*diag([0,1,0])*q_B_center);
s3=trace(q_A_center'*rotation*diag([0,0,1])*q_B_center)/trace(q_B_center'*diag([0,0,1]) *q_B_center);
rotation=U*V';
scale =diag([s1,s2,s3])
translation =centroid_q_A-rotation*scale*centroid_q_B;
```


## Appendix 2:. Calculate rotation and scale of coplanar point sets in arbitrary alignment

Two coplanar point sets have rank deficiency to determine $S$ (rank $=3$ ) together with arbitrary 3D rotation. However, one can determine unique $S$ together with unique $R_{x}, R_{y}, R_{z}$ giving one of the scale factors. $s_{i}=k$

As follow.
Giving $R_{A}$ and $R_{B}$ the rotation matrices that transform $A$ and $B$ into 2D point sets on x-y plane; $\bar{A}$ and $\bar{B}$, respectively. The transformation of point sets into $x-y$ plane could be determined using with Rodrigues' Rotation Formula.

According to (5),
$\overline{A^{\prime}}=R_{A} R S R_{B}^{T} \bar{B}$
Where $\overline{A^{\prime}}$ is transform of $B$ on the same plane at the closest to. $\bar{A}$
Giving $P=R_{A} R S R_{B}^{T}$
We can determine $P_{2 \times 2}$ by calculate (13) at rank $=2$.
Since $\overline{A^{\prime}}$ is 2D data on x-y plane, $P_{3,1}, P_{3,2}=0$ as follow,
$P=\left[\begin{array}{cc}P_{2 \times 2} & x_{1} \\ & x_{2} \\ 0 & 0\end{array} x_{3}\right]$
$x_{1}, x_{2}, x_{3}$ in (A3) could be determined using
$\overline{A^{\prime}}=P R_{B} B$
Giving SVD of $P=\widetilde{U} \widetilde{\Sigma} \widetilde{V}^{T}$. (A2) and (A4) denotes,
$\widetilde{V}=R_{B}$ and $\widetilde{\Sigma}=S$
Multiply $P$ with its transverse
$P^{T} P=\widetilde{V} S^{2} \widetilde{V}^{T}$
According to (A6), each column of $\widetilde{V}$ represent the eigenvector of $P^{T} P$ with $s_{i}$, the corresponding eigenvalue.
$x_{1}, x_{2}, x_{3}$ could be determined by solving the eigenvalue and eigenvector problem
$P^{T} P R_{B}(:, 3)=k R_{B}(:, 3)$

## References

[1] Chatrasingh M, Suthakorn J. A novel design of N-fiducial phantom for automatic ultrasound calibration. J Med Phys 2019;44:191-200.
[2] Jiang WW, Zhou GQ, Lai KL, et al. A fast 3-D ultrasound projection imaging method for scoliosis assessment. Math Biosci Eng 2019;16:1067-81.
[3] Xu HH, Gong YC, Xia XY, et al. Gabor-based anisotropic diffusion with lattice Boltzmann method for medical ultrasound despeckling. Mathemat Biosci Eng: MBE 2019;16:7546-61
[4] Suthakorn J, Tanaiutchawoot N, Wiratkapan C, Ongwattanakul S. Breast biopsy navigation system with an assisted needle holder tool and 2D graphical user interface. Europ J Radiol Open 2018;5:93-101.
[5] Ouyang Y, Zhou Z, Wu W, et al. A review of ultrasound detection methods for breast microcalcification. Math Biosci Eng 2019;16:1761-85.
[6] Suthakorn J, Tanaiutchawoot N, Wiratkapan C. Ultrasound calibration with ladder phantom at multiple depths for breast biopsy navigation system. Theor Appl Mech Lett 2020;10(5):343-53.
[7] Jung EM, Friedrich C, Hoffstetter P, Dendl LM, Klebl F, Agha A, et al. Volume navigation with contrast enhanced ultrasound and image fusion for percutaneous interventions: first results. PLoS One 2012;7(3):e33956.
[8] Machado I, Toews M, George E, Unadkat P, Essayed W, Luo J, et al. Deformable MRI-Ultrasound registration using correlation-based attribute matching for brain shift correction: Accuracy and generality in multi-site data. Neuroimage 2019;202: 116094.
[9] Buckner CA, Venkatesan A, Locklin JK, Wood BJ. Real-time Sonography with Electromagnetic Tracking Navigation for Biopsy of a Hepatic Neoplasm Seen Only on Arterial Phase Computed Tomography. J Ultrasound Med 2011;30:253-6.
[10] Maiseli B, Gu Y, Gao H. Recent developments and trends in point set registration methods. J Vis Commun Image Represent 2017;46:95-106.
[11] Arun KS, Huang TS, Blostein SD. Least-squares fitting of two 3-D point sets. IEEE Trans Pattern Anal Mach Intell 1987;5:698-700.
[12] Horn BK. Closed-form solution of absolute orientation using unit quaternions. Josa a 1987;4:629-42.
[13] Gower J.C. G.B. Dijksterhuis, Procrustes problems, Oxford University Press, 2011.
[14] Bennani Dosse M, Ten Berge J. Anisotropic orthogonal procrustes analysis. J Classif 2010;27(1):111-28.
[15] Bennani Dosse M, Kiers HAL, Ten Berge JMF. Anisotropic generalized Procrustes analysis. Comput Stat Data Anal 2011;55(5):1961-8.
[16] Shi X, Peng J, Li J, Yan P, Gong H. The iterative closest point registration algorithm based on the normal distribution transformation. Procedia Comput Sci 2019;147: 181-90.
[17] Chen ECS, McLeod AJ, Baxter JSH, Peters TM. Registration of 3D shapes under anisotropic scaling. Int J Comput Assist Radiol Surg 2015;10(6):867-78.
[18] Maier-Hein L, dos Santos TR, Franz AM, Meinzer HP. Iterative Closest Point Algorithm in the Presence of Anisotropic Noise. Bildverarbeitung für die Medizin 2010;2010:4322..
[19] Qu J, Gong L, Yang L. A 3D point matching algorithm for affine registration. Int J Comput Assist Radiol Surg 2011;6(2):229-36.
[20] Umeyama S. Least-squares estimation of transformation parameters between two point patterns. IEEE Trans Pattern Anal Mach Intell 1991;13:376-80.
[21] Bellman R. Introduction to matrix analysis. Philadelphia: Society for Industrial and Applied Mathematics; 1997.
[22] Gonçalves PJS, Torres PMB, Santos F, António R, Catarino N, Martins JMM. A vision system for robotic ultrasound guided orthopaedic surgery. J Intell Rob Syst 2015;77(2):327-39.
[23] Michael Fitzpatrick J. Fiducial registration error and target registration error are uncorrelated. Proc. SPIE 7261, Medical Imaging 2009: Visualization, ImageGuided Procedures, and Modeling, 726102.
[24] Sato Y, Nakamoto M, Tamaki Y, Sasama T, Sakita I, Nakajima Y, et al. Image guidance of breast cancer surgery using 3-D ultrasound images and augmented reality visualization. IEEE Trans Med Imaging 1998;17(5):681-93.
[25] Maier-Hein L, Mountney P, Bartoli A, Elhawary H, Elson D, Groch A, et al. Optical techniques for 3D surface reconstruction in computer-assisted laparoscopic surgery. Med Image Anal 2013;17(8):974-96.
[26] Kapsalyamov A, Hussain S, Jamwal PKA. novel compliant surgical robot: Preliminary design analysis. Math Biosci Eng 2020;17:1944-58.
[27] Fitzgibbon AW. Robust registration of 2D and 3D point sets. Image Vis Comput 2003;21(13-14):1145-53.
[28] Fu Y, Lei Y, Wang T, Curran WJ, Liu T, Yang X. Deep learning in medical image registration: a review. Phys Med Biol 2020;65:20TR01.
[29] Lu Y, Gao K, Zhang T, Xu T, Tang D. A novel image registration approach via combining local features and geometric invariants. PLoS One 2018;13(1): e0190383.
[30] Xu G, Zheng A, Li X, Su J. A method to calibrate a camera using perpendicularity of 2D lines in the target observations. Sci Rep 2016;6:1-15.
[31] Hachadorian RL, Bruza P, Jermyn M, Gladstone DJ, Pogue BW, Jarvis LA. Imaging radiation dose in breast radiotherapy by X-ray CT calibration of Cherenkov light. Nat Commun 2020;11(1).
[32] Ilbey S, Top CB, Güngör A, Çukur T, Saritas EU, Güven HE. Fast System Calibration with Coded Calibration Scenes for Magnetic Particle Imaging. IEEE Trans Med Imaging 2019;38:2070-80.
[33] Zheng X, Li Z, Chun X, Yang X, Liu K. A model-based method with geometric solutions for gaze correction in eye-tracking. Math Biosci Eng 2020;17:1396-412.
[34] Piga NA, Onyshchuk Y, Pasquale G, Pattacini U, Natale L. ROFT: Real-Time Optical Flow-Aided 6D Object Pose and Velocity Tracking. IEEE Rob Autom Lett 2022;7: 159-66.
[35] Sattayasoonthorn P, Suthakorn J, Chamnanvej S, Miao J, Kottapalli AGP. LCP MEMS implantable pressure sensor for Intracranial Pressure measurement. In: In

The 7th IEEE International Conference on Nano/Molecular MedicIne and EngIneerIng; 2013. p. 63-7.
[36] Neatpisarnvanit C, Suthakorn J. Intramedullary nail distal hole axis estimation using Blob analysis and Hough transform. IEEE Conference on Robotics, Automation, and Mechatronics 2006:1-6.
[37] Gul F, Rahiman W, Nazli Alhady SS, Chen K. A comprehensive study for robot navigation techniques. Cogent Eng 2019;6(1).
[38] Mingachev E, Lavrenov R, Tsoy T, et al. Comparison of ros-based monocular visual slam methods: Dso, 1dso, orb-slam2 and dynaslam. In International Conference on Interactive Collaborative Robotics 2020:222-33.
[39] Cipresso P, Giglioli IAC, Raya MA, Riva G. The past, present, and future of virtual and augmented reality research: a network and cluster analysis of the literature. Front Psychol 2018;9:2086.
[40] Li C, Sun X, Li Y. Information hiding based on Augmented Reality. Math Biosci Eng 2019;16:4777-87.


[^0]:    * Corresponding author.

    E-mail addresses: maria@bartlab.org (M. Chatrasingh), cholatip.wir@mahidol.ac.th (C. Wiratkapun), jackrit.sut@mahidol.ac.th, jackrit@bartlab.org (J. Suthakorn).

